Fermion Pair Production From an Electric Field Varying in Two Dimensions

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Abstract

The Hamiltonian describing fermion pair production from an arbitrarily time-varying electric field in two dimensions is studied using a group-theoretic approach. We show that this Hamiltonian can be encompassed by two, commuting $SU(2)$ algebras, and that the two-dimensional problem can therefore be reduced to two one-dimensional problems. We compare the group structure for the two-dimensional problem with that previously derived for the one-dimensional problem, and verify that the Schwinger result is obtained under the appropriate conditions.

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I. INTRODUCTION

Fermion pair production takes place in a large number of physical situations; a comprehensive review of its applications in atomic, nuclear, elementary particle physics, astrophysics and cosmology is given in Ref. [1]. Consequently, the problem of pair production from classical external electric fields has been the subject of considerable theoretical interest.[2]-[17] The rate of fermion pair production from a uniform, static electric field was originally calculated by Schwinger [2] to be

$$\omega = \frac{\alpha E^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \exp\left(-\frac{n\pi m^2}{|eE|} \right) \quad , \tag{1}$$

where m is the mass of the produced fermions. To date, an analytic formalism that successfully addresses the general problem of fields which vary arbitrarily in both time and space has not been developed. However, numerous approaches have been suggested which address particular special cases. We previously discussed an approach for predicting the rate of pair production from a spatially homogeneous but arbitrarily time-varying field, provided the field is constrained to point in a fixed direction.[17] We now investigate an extension to this formalism that allows for a field varying in two dimensions.

We begin by writing the interaction Hamiltonian for fermions in an electric field. We adopt the gauge

$$A_0 = 0 \ , A_i = -\int_{-\infty}^{t} E_i(t)\, dt \quad . \tag{2}$$

The interaction Hamiltonian is then given by

$$H_I = -eA_i \int d^3x\, \overline{\psi}_{in}(x) \gamma_i \psi_{in}(x) \quad , \tag{3}$$

where γ_i is the i^{th} 4×4 Dirac γ-matrix, and summation over like indices is assumed. The incoming Dirac field, ψ_{in}, is that of free fermions,

$$\psi_{in} = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \sqrt{\frac{m}{k_0}} \sum_{\beta} \left[b_\beta(k) u_\beta(k) e^{-ik\cdot x} + d^\dagger_\beta(\tilde{k}) v_\beta(\tilde{k}) e^{ik\cdot x} \right] \quad , \tag{4}$$

where b and $d^\dagger$ are the usual fermion creation and antifermion annihilation operators, and u and v are the two-component fermion and antifermion spinors. We use the symbol k to denote the four-vector $(k_0, \mathbf{k})$ and $\tilde{k}$ to denote $(k_0, -\mathbf{k})$; k represents the initial momentum of the fermions, and is a time-independent quantity. The mass of the created fermions is given by m, and their charge by e.

We chose the one-dimensional configuration as a starting point because, if the field varies in only one direction, the Hamiltonian contains only one Dirac γ-matrix, and an $SU(2)$ algebra is sufficient to encompass the Hamiltonian. In the two-dimensional case, the Hamiltonian contains two Dirac γ-matrices. The appropriate algebra is then an $SO(4)$ algebra, which is isomorphic to two commuting $SU(2)$ algebras, as we illustrate below.

II. PAIR EMISSION FROM A TWO-DIMENSIONAL ELECTRIC FIELD

For an electric field that varies in the plane defined by the directions $i = 1, 2$, the interaction picture Hamiltonian is

$$H = \int d^3\mathbf{k} \left\{ \left(2k_0 - 2e\frac{A_i k_i}{k_0} \right) J_0(k) - \frac{e\mu_i A_i}{k_0} \left[J_+^{(i)}(k) + J_-^{(i)}(k) \right] \right\} \quad , \tag{5}$$

where summation of i over indices 1 and 2 is implied. μ_i in this expression is defined as $\mu_i = \sqrt{k_0^2 - k_i^2}$. We have defined the operators in analogy to

the one-dimensional case:[17]

$$
\begin{aligned}
J_+^{(i)} &= \frac{m}{\mu_i}\sum_{\alpha\beta} b_\alpha^\dagger(k) d_\beta^\dagger(\tilde{k}) \bar{u}_\alpha(k)\gamma_i v_\beta(\tilde{k}) \quad , \qquad i=1,2 \\
J_-^{(i)} &= \left[J_+^{(i)}\right]^\dagger \quad , \qquad i=1,2 \\
J_0 &= \frac{1}{2}\sum_\alpha \left[b_\alpha^\dagger(k) b_\alpha(k) - d_\alpha(\tilde{k}) d_\alpha^\dagger(\tilde{k})\right] \quad .
\end{aligned} \tag{6}
$$

With an additional operator,

$$
Q = \sum_{\alpha\beta} \left[b_\alpha^\dagger b_\beta \bar{u}_\alpha \gamma_3\gamma_5 u_\beta + d_\alpha d_\beta^\dagger \bar{v}_\alpha \gamma_3\gamma_5 v_\beta\right] \quad , \tag{7}
$$

these operators form an $SO(4)$ algebra.

From linear combinations of these operators, we can form two commuting $SU(2)$ algebras, which we denote by $I_{+,-,0}$ and $T_{+,-,0}$, as follows:

$$
\begin{aligned}
I_+ &= a\,J_+^{(1)} + a^*\,J_+^{(2)} \quad , \\
I_- &= [I_+]^\dagger \quad , \\
I_0 &= \frac{1}{2}\,J_0 + \frac{m}{4\mu}\,Q \quad ,
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
T_+ &= a^*\,J_+^{(1)} + a\,J_+^{(2)} \quad , \\
T_- &= [T_+]^\dagger \quad , \\
\text{and} \qquad T_0 &= \frac{1}{2}\,J_0 - \frac{m}{4\mu}\,Q \quad ,
\end{aligned} \tag{9}
$$

where

$$
a = \sqrt{\frac{\mu_1\mu_2}{8(\mu_1\mu_2\,-\,k_1k_2)}} + i\sqrt{\frac{\mu_1\mu_2}{8(\mu_1\mu_2\,+\,k_1k_2)}} \quad , \tag{10}
$$

and we have defined $\mu = \sqrt{k_0^2 - k_1^2 - k_2^2}$. Each of these $SU(2)$ algebras is in the $j = \frac{1}{2}$ representation.

The group-theoretic approach has been previously discussed by Perelemov[18], but the algebras we have derived are distinct from the algebras he considered. The $SU(2)$ algebras utilized in Ref.[18] are constructed from the Dirac

γ-matrices, whereas the $SU(2)$ commutation relations of the operators in Eqs. 8 and 9 follow from the completeness and orthogonality of the Dirac spinors $u(p)$ and $v(p)$.

The Hamiltonian in Eq. 5 can be rewritten as a linear combination of elements of these two $SU(2)$ algebras, and diagonalized via a Bogoliubov [19] transformation. The Bogoliubov transformation takes the usual form:

$$\tilde{b}_\alpha(k) = \mathcal{U}_{\alpha\beta}(k) b_\beta(k) + \mathcal{V}_{\alpha\beta}(k) d^\dagger_\beta(\tilde{k}) \tag{11}$$

$$\tilde{d}_\alpha(\tilde{k}) = \mathcal{X}_{\alpha\beta}(k) d_\beta(\tilde{k}) + \mathcal{Y}_{\alpha\beta}(k) b^\dagger_\beta(k) \quad , \tag{12}$$

where the coefficients are time-dependent, 2×2 matrices. Requiring that the transformation preserve the canonical commutation relations constrains the coefficients to satisfy the relations:

$$\mathcal{U}\mathcal{U}^\dagger + \mathcal{V}\mathcal{V}^\dagger = 1 \quad , \quad \mathcal{X}\mathcal{X}^\dagger + \mathcal{Y}\mathcal{Y}^\dagger = 1 \quad ,$$

$$\mathcal{U}\mathcal{V}^T + \mathcal{V}\mathcal{X}^T = 0 \quad .$$

We further require that the Bogoliubov transformation yield the diagonal Hamiltonian,

$$H = \int d^s\mathbf{k}\ \varepsilon(k) \sum_{\alpha\beta} \left[\tilde{b}^\dagger_\alpha(k) \tilde{b}_\alpha(k) - \tilde{d}_\alpha(\tilde{k}) \tilde{d}^\dagger_\alpha(\tilde{k}) \right] \quad , \tag{13}$$

where ε is the total energy,

$$\varepsilon = \sqrt{m^2 + (\mathbf{k} - e\mathbf{A})^2} \quad .$$

This requirement constrains the coefficients to be:

$$\mathcal{U} = \cos\theta\ I \quad , \tag{14}$$

$$\text{and} \qquad \mathcal{Z} = \mathcal{U}^{-1}\mathcal{V} = \frac{-mA_i \tan\theta}{\sqrt{k_0^2 \mathbf{A}^2 - (\mathbf{A}\cdot\mathbf{k})^2}} \left[\bar{u}_\alpha \gamma_i v_\beta \right] \quad , \tag{15}$$

and summation over i is implied. θ is defined by the relations

$$\cos\theta = \sqrt{\frac{\varepsilon+\rho}{2\varepsilon}} \qquad , \qquad \sin\theta = -\sqrt{\frac{\varepsilon-\rho}{2\varepsilon}} \quad , \tag{16}$$

and ρ is defined by

$$\rho = k_0 - \frac{e\mathbf{A}\cdot\mathbf{k}}{k_0} \quad . \tag{17}$$

Alternatively, one can write the Bogoliubov transformation as a linear operator,

$$R = R_1 R_2 \quad , \tag{18}$$

such that

$$\tilde{b} = \quad R(t)\, b\, R^{\dagger}(t) \quad . \tag{19}$$

R_1 and R_2 can each be written in terms of only one of the two commuting SU(2) algebras:

$$\begin{aligned} R_1 &= \exp\left[\eta\ I_+\right]\exp\left[\log(1+|\eta|^2)\ I_0\right]\exp\left[-\eta^*\ I_-\right] \\ R_2 &= \exp\left[\eta^*\ T_+\right]\exp\left[\log(1+|\eta|^2)\ T_0\right]\exp\left[-\eta\ T_-\right] \quad , \end{aligned} \tag{20}$$

where

$$\eta = \frac{-1}{(a^2 - a^{*2})}\left[\frac{\tan\theta\,(a\mu_1 A_1 - a^*\mu_2 A_2)}{\sqrt{k_0^2\mathbf{A}^2 - (\mathbf{A}\cdot\mathbf{k})^2}}\right] \tag{21}$$

gives the desired diagonal Hamiltonian.

Writing the Bogoliubov transformation as a linear operator is useful when calculating the rate of pair production from an electric field. The physical vacuum after the field has been turned off ($t > T$), $|Z(T)\rangle$, is related to the vacuum before the field was turned on ($t < -T$), $|0_i\rangle$, by

$$|Z(T)\rangle = R(T)|0_i\rangle \quad . \tag{22}$$

Similarly, the physical creation operators for fermions and anti-fermions at times $t > T$ are $\tilde{b}^\dagger(k)$ and $\tilde{d}^\dagger(k)$ respectively.[17] The probability amplitude of producing no pairs, S_0, is therefore given by

$$\begin{aligned} S_0 &= \lim_{t\to\infty} \langle Z|U_I|0_i\rangle \\ &= \langle 0_i|\tilde{U}|0_i\rangle \quad , \end{aligned} \tag{23}$$

where $\tilde{U} = R^\dagger U_I$. One can solve for $\tilde{U}$ directly, through

$$i\frac{d\tilde{U}}{dt} = \tilde{H}\tilde{U} \quad ,$$

where

$$\tilde{H} = R^\dagger H R - iR^\dagger \dot{R} \quad . \tag{24}$$

The probability amplitude for producing no pairs is related to the rate of pair production, ω, by

$$|S_0|^2 = e^{-\int \omega \; d^4x} \quad . \tag{25}$$

The operator R_1 has the matrix representation

$$R_1 = \begin{pmatrix} \cos\frac{\alpha_1}{2} & \sin\frac{\alpha_1}{2}e^{-i\gamma_1} \\ -\sin\frac{\alpha_1}{2}e^{-i\gamma_1} & \cos\frac{\alpha_1}{2} \end{pmatrix} \quad , \tag{26}$$

where $\eta = \tan\frac{\alpha}{2}e^{-i\gamma}$. R_2 is defined analogously. One observes that, to satisfy Eq. 20 , $\alpha_1 = \alpha_2$ and $\gamma_1 = -\gamma_2$; the subscripts on α and γ are therefore dropped. The matrix representation allows one to easily calculate the Hamiltonian $\tilde{H}$. By substituting Eqs. 8 and 9 into the Hamiltonian of Eq. 5, one can show that this Hamiltonian can be written $H = H_1 + H_2$, where

$$\begin{aligned} H_1 &= \int d^3k \quad \left\{ \left(2k_0 - \frac{2eA_ik_i}{k_0}\right) I_0 \right. \\ &\left. \quad + 2ie\mu \left[\left(\frac{aA_1}{\mu_2} - \frac{a^*A_2}{\mu_1}\right) I_+ - \left(\frac{a^*A_1}{\mu_2} - \frac{aA_2}{\mu_1}\right) I_- \right] \right\} \end{aligned} \tag{27}$$

and

$$
\begin{aligned}
H_2 \quad = \smallint d^3k \quad & \left\{ \left(2k_0 - \frac{2eA_ik_i}{k_0} \right) T_0 \right. \\
& \left. + \, 2ie\mu \quad \left[\left(\frac{-a^*A_1}{\mu_2} + \frac{aA_2}{\mu_1} \right) T_+ + \left(\frac{aA_1}{\mu_2} - \frac{a^*A_2}{\mu_1} \right) T_- \right] \right\} \quad (28)
\end{aligned}
$$

It follows that $\tilde{H}$ can be written $\tilde{H} = \tilde{H}_1 + \tilde{H}_2$, where

$$
\tilde{H}_1 = R_1^\dagger H_1 R_1 - iR_1^\dagger \dot{R}_1 \tag{29}
$$

and

$$
\tilde{H}_2 = R_2^\dagger H_2 R_2 - iR_2^\dagger \dot{R}_2 \quad . \tag{30}
$$

The explicit expressions for $\tilde{H}_1$ and $\tilde{H}_2$ are then

$$
\begin{aligned}
\tilde{H}_1 \quad = \quad & \int d^3k \, \left\{ \left[2\varepsilon(k) \, + \, 2(\dot{\gamma} \sin^2 \frac{\alpha}{2}) \right] I_0 \right. \\
& \left. -\frac{1}{2} \left[(i\dot{\alpha} + \dot{\gamma} \sin\alpha) e^{-i\gamma} I_+ \, + (-i\dot{\alpha} \, + \, \dot{\gamma} \sin\alpha) e^{i\gamma} I_- \right] \right\} \quad (31)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{H}_2 \quad = \quad & \int d^3k \, \left\{ \left[2\varepsilon(k) \, - \, 2(\dot{\gamma} \sin^2 \frac{\alpha}{2}) \right] T_0 \right. \\
& \left. -\frac{1}{2} \left[(i\dot{\alpha} - \dot{\gamma} \sin\alpha) e^{i\gamma} T_+ \, + (-i\dot{\alpha} \, - \, \dot{\gamma} \sin\alpha) e^{-i\gamma} T_- \right] \right\} \quad . \, (32)
\end{aligned}
$$

If we now write $\tilde{U}$ as a product, $\tilde{U} = \tilde{U}_1 \tilde{U}_2$, where $\tilde{U}_1$ and $\tilde{U}_2$ are each written in the most general form of an element of the respective $SU(2)$ groups:

$$
\begin{aligned}
\tilde{U}_1 \quad = \quad & \exp\left[-i \int d^3\mathbf{k} \; \phi_1 I_0 \right] \exp\left[\int d^3\mathbf{k} \; \tau_1 I_+ \right] \\
& \times \, \exp\left[\int d^3\mathbf{k} \; \log(1 + |\tau_1|^2) \, I_0 \right] \, \exp\left[- \int d^3\mathbf{k} \; \tau_1^* I_- \right] \quad (33)
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{U}_2 \quad = \quad & \exp\left[-i \int d^3\mathbf{k} \; \phi_2 T_0 \right] \exp\left[\int d^3\mathbf{k} \; \tau_2 T_+ \right] \\
& \times \, \exp\left[\int d^3\mathbf{k} \; \log(1 + |\tau_2|^2) \, T_0 \right] \, \exp\left[- \int d^3\mathbf{k} \; \tau_2^* T_- \right] , (34)
\end{aligned}
$$

then the differential equation for $\tilde{U}$,

$$i\frac{d\tilde{U}}{dt} = \tilde{H}\tilde{U} \quad ,$$

separates into two independent equations for $\tilde{U}_1$ and $\tilde{U}_2$. We show this as follows:

$$i\frac{d\tilde{U}}{dt} = i\frac{d\tilde{U}_1}{dt}\tilde{U}_2 + i\tilde{U}_1\frac{d\tilde{U}_2}{dt} = \left(\tilde{H}_1 + \tilde{H}_2\right)\tilde{U}_1\tilde{U}_2 \quad . \tag{35}$$

This is the sum of the two equations:

$$\left[i\frac{d\tilde{U}_1}{dt} = \tilde{H}_1\tilde{U}_1\right]\tilde{U}_2 \tag{36}$$

and

$$\tilde{U}_1\left[i\frac{d\tilde{U}_2}{dt} = \tilde{H}_2\tilde{U}_2\right] \quad . \tag{37}$$

These are independent differential equations for $\tilde{U}_1$ and $\tilde{U}_2$, each equation containing elements of only one $SU(2)$ algebra. When we insert our ansatz for $\tilde{U}_1$ and $\tilde{U}_2$ into the above differential equations, we obtain differential equations for the coefficients τ_1, τ_2, ϕ_1 and ϕ_2. One can proceed to solve these differential equations in precisely the same manner as in the one-dimensional case.[17]

Let

$$z = \tau\exp(-i\phi + i\gamma) \quad . \tag{38}$$

The resulting differential equation for z_1 is

$$\begin{aligned} i\dot{z}_1 \quad = \quad & -\frac{1}{2}(\dot{\gamma}\sin\alpha + i\dot{\alpha}) + \\ & +2\left[\varepsilon(k) + \dot{\gamma}(\sin^2\frac{\alpha}{2} + \frac{1}{2})\right]z_1 \\ & +\frac{1}{2}(\dot{\gamma}\sin\alpha - i\dot{\alpha})z_1^2 \quad . \end{aligned} \tag{39}$$

The corresponding differential equation for z_2 is

$$\begin{aligned} i\dot{z}_2 &= -\frac{1}{2}(-\dot{\gamma}\sin\alpha + i\dot{\alpha}) + \\ &+2\left[\varepsilon(k) - \dot{\gamma}(\sin^2\frac{\alpha}{2} - \frac{1}{2})\right] z_2 \\ &+\frac{1}{2}(-\dot{\gamma}\sin\alpha - i\dot{\alpha})z_2^2 \quad . \end{aligned} \tag{40}$$

One can write $\dot{\alpha}$ and $\dot{\gamma}$ explicitly, in terms of the electric field and vector potential. The expression for $\dot{\alpha}$ is:

$$\dot{\alpha} = \left(\frac{e}{\varepsilon^2}\right) \frac{[(\mathbf{k} - e\mathbf{A}) \times \mathbf{E}] \cdot (\mathbf{k} \times \mathbf{A}) + m^2(\mathbf{E} \cdot \mathbf{A})}{\sqrt{k_0^2\mathbf{A}^2 - (\mathbf{A} \cdot \mathbf{k})^2}} \quad , \tag{41}$$

with $\dot{\alpha} = 0$ at $t = -\infty$. The expression for $\dot{\gamma}$ is:

$$\dot{\gamma} = -\frac{\mu k_0 |\mathbf{A} \times \mathbf{E}|}{k_0^2\mathbf{A}^2 - (\mathbf{A} \cdot k)^2} \quad . \tag{42}$$

Note that only $\dot{\gamma}$ appears explicitly in the equations; when calculating $|\tau|$ (see Eq. 43 below), the initial condition on γ is irrelevant. When $\mathbf{A}$ and $\mathbf{E}$ are parallel ($\dot{\gamma} = 0$), Eqs. 39 and 40 reduce to the corresponding equation calculated for the one-dimensional case.[17] When $\dot{\gamma} \neq 0$, the two-dimensional nature of the equations is manifested.

Finally, one can calculate the rate of pair production, ω, via S_0. The above definitions give us:

$$\begin{aligned} S_0 = \langle 0_i|\tilde{U}|0_i\rangle &= \exp\left[i\int d^3\mathbf{k}\,(\frac{\phi_1 + \phi_2}{2})\right] \\ &\times \exp\left[-\int d^3\mathbf{k}\,\log\sqrt{(1 + |\tau_1|^2)(1 + |\tau_2|^2)}\right] \end{aligned} \tag{43}$$

Then

$$|S_0|^2 = |\langle 0_i|\tilde{U}|0_i\rangle|^2 = e^{-\int d^3\mathbf{k}\,\log[(1+|z_1|^2)(1+|z_2|^2)]} \quad . \tag{44}$$

Once the differential equations have been solved for z_1 and z_2 (in general, this must be done numerically), the rate of pair production is easily obtained.

To verify this approach, consider the very simple example of a uniform, static electric field which is oriented at an angle θ to the x-axis. Then $|\mathbf{E} \times \mathbf{A}| = 0$, and $\dot{\gamma} = 0$. In this case, $\dot{\alpha}$ reduces to

$$\dot{\alpha} = \frac{e k_\perp E_0}{\varepsilon^2} \quad , \tag{45}$$

where

$$k_\perp = \sqrt{k_0^2 - k_\parallel^2} = \sqrt{k_0^2 - \frac{(\mathbf{k} \cdot \mathbf{E})^2}{\mathbf{k}^2}} \quad . \tag{46}$$

With these values, the expressions for $i\dot{z}_1$ and $i\dot{z}_2$ are identical, and each is equal to the expression which applied in the one-dimensional case.[17] With $z_1 = z_2$, Eq. 44 reduces to

$$|S_0|^2 = \ |\langle 0_i|\tilde{U}|0_i\rangle|^2 = \ e^{-2\int d^3\mathbf{k}\ \log(1+|z_1|^2)} \quad , \tag{47}$$

which again is identical to the one-dimensional result.[17] In this case, the rate of pair production has been shown to be equal to that calculated by Schwinger, given in Eq. 1.

III. CONCLUSIONS

We have shown that the Hamiltonian describing fermion pair production from an arbitrarily time-varying electric field in two dimensions is encompassed by an $SO(4)$ algebra. We have also explicitly constructed the two commuting $SU(2)$ algebras in the direct product $SU(2) \times SU(2)$, which is isomorphic to this $SO(4)$ algebra. The one-dimensional problem is described by an $SU(2)$ algebra in the $j = 1$ representation, while the two-dimensional

problem is described by two $SU(2)$ algebras in the $j = \frac{1}{2}$ representation. However, when the one-dimensional problem and the two-dimensional problem are each considered in the lowest-dimensional representation, one sees that the off-diagonal elements are real in the one-dimensional case and complex in the two-dimensional case. The extra degree of freedom present in the two-dimensional case is manifested in this way. Indeed, it can easily be shown that the factor γ in Eq. 38 is the Berry's phase.

This group-theoretic approach may simplify the calculation of the rate of fermion pair production from the field, since the two-dimensional problem can in this way be reduced to two one-dimensional problems. We verify our approach by showing that the Schwinger formula for pair production can be obtained for the special case of a uniform, static electric field.

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